Version of 30 July 1999

Nonamenable Products are not Treeable

by Robin Pemantle and Yuval Peres

Abstract. Let X and Y be infinite graphs, such that the automorphism group of X is nonamenable, and the automorphism group of Y has an infinite orbit. We prove that there is no automorphism-invariant measure on the set of spanning trees in the direct product $X \times Y$. This implies that the minimal spanning forest corresponding to i.i.d. edge-weights in such a product, has infinitely many connected components almost surely.

§1. Introduction.

There are several natural ways to pick a random spanning tree in a finite graph, notably the *uniform spanning tree* and the *minimal spanning tree* (for random edge weights). The limits of these models on infinite graphs sometimes give spanning *forests*, rather than trees. In this note, we present a large class of graphs where one cannot pick a random subtree in an automorphism-invariant manner.

Definition. A subtree $\Gamma = (V_\Gamma, E_\Gamma)$ in a graph $X = (V_X, E_X)$ is an acyclic connected graph, with $V_\Gamma \subset V_X$ and $E_\Gamma \subset E_X$. If $V_\Gamma = V_X$, then Γ is called a *spanning* tree of X. By identifying a subtree with the indicator function of its edges, we may view the ensemble of spanning trees in X as a Borel set in the compact metrizable space $\{0,1\}^{E_X}$.

Theorem 1.1. *Let X and Y be infinite, locally finite graphs. Suppose that $G \subset \mathrm{Aut}(X)$ is a closed nonamenable subgroup of $\mathrm{Aut}(X)$, and $H \subset \mathrm{Aut}(Y)$ has an infinite orbit. Then there is no $G \times H$-invariant probability measure on the set of spanning trees of the direct product graph $X \times Y$.*

Given two graphs $X = (V_X, E_X)$ and $Y = (V_Y, E_Y)$, the **direct product graph** $X \times Y$ has vertex set $V_X \times V_Y$; the vertices (x_1, y_1) and (x_2, y_2) in $V_X \times V_Y$ are taken to be adjacent in $X \times Y$ iff either $x_1 = x_2$ and $[y_1, y_2] \in E_Y$, or $y_1 = y_2$ and $[x_1, x_2] \in E_X$.

1991 Mathematics Subject Classification. Primary 60B99. Secondary 60D05, 20F32 .
Key words and phrases. Cayley graphs, amenability, spanning trees.
Research partially supported NSF grant DMS-9803597.

In the next section we provide some probabilistic motivation for Theorem 1.1, by describing its application to *minimal spanning trees* (corresponding to i.i.d. weights) in finite graphs, and their limits, *minimal spanning forests* in infinite graphs.

The following extension of Theorem 1.1 is sometimes useful.

COROLLARY 1.2. *Under the hypothesis of Theorem 1.1, there is no $G \times H$-invariant probability measure on the set of subtrees of the direct product graph $X \times Y$.*

Proof. ¿From the assumption that Γ is a random subtree in $X \times Y$ with a $G \times H$-invariant law, we will obtain a contradiction. Let $\mathcal{S}(\Gamma, n)$ denote the set of vertices in $X \times Y$ at graphical distance n from Γ. For every $n \geq 1$ and every $v \in \mathcal{S}(\Gamma, n)$, choose uniformly at random a neighbor v' of v such that $v' \in \mathcal{S}(\Gamma, n-1)$, and add the edge $[v, v']$ to Γ. This yields a random spanning tree in $X \times Y$ with a $G \times H$-invariant law, and contradicts Theorem 1.1.

∎

Next, we present a variant of Theorem 1.1 that does not require any apriori graph structure.

Definitions. (i) A spanning tree in a countable set Υ is an acyclic connected graph with vertex set Υ; it need not be locally finite. We may view the ensemble of spanning trees in Υ as a Borel set in the compact metrizable space $\{0,1\}^{\Upsilon \times \Upsilon}$.

(ii) A countable group G is **treeable** if there exists a Borel probability measure on the set of spanning trees in G, which is invariant under the action of G by right multiplication.

Obviously, a free group is treeable. The work of Ornstein and Weiss (1987) implies that any discrete amenable group is treeable; see Theorem 5.3 of Benjamini, Lyons, Peres and Schramm (1999a), denoted BLPS (1999a) below, for an alternative proof. On the other hand, results of Adams and Spatzier (1990) imply that groups with Kazhdan's property T are not treeable. The following theorem exhibits simpler examples of nontreeable groups.

THEOREM 1.3. *Let G and H be countably infinite groups. If the direct product $G \times H$ is treeable, then G and H are amenable.*

In particular, the direct product of a free group (on two or more generators) with any infinite group is nontreeable. It is an intriguing unsolved problem to find a geometric characterization for treeable groups.

Theorem 1.3 seems close in spirit to results of Adams (1988, 1994); however, it appears that there is no direct implication from one result to the other (S. Adams, R. Lyons and B. Weiss, private communications). The referee has suggested that perhaps the *techniques* of Adams (1988) could be used to give an alternative proof of our main results.

After seeing an earlier version of the present paper, R. Lyons has pointed out that Theorem 1.3 (but not Theorem 1.1) could also be inferred from recent work of Gaboriau (1998) on the "cost" of equivalence relations.

In the next section we discuss minimal spanning trees and forests. Section 3 contains background material on amenability. Finally, in Section 4, we establish a common generalization of Theorem 1.3 and Theorem 1.1.

§2. Minimal spanning trees and forests.

Suppose that the edges of a connected finite graph (V, E) are labeled with i.i.d. random variables $\{U_e\}_{e \in E}$, that are uniform in $[0, 1]$. The corresponding **minimal spanning tree** Γ (the spanning tree that minimizes the sum of labels on its edges) consists of edges e such that there is no path between the endpoints of e where all edges on the path have labels lower than U_e.

If we now consider a sequence of connected finite graphs X_n that exhaust an infinite, locally finite connected graph X, then the corresponding minimal spanning trees Γ_n converge almost surely to a random spanning subgraph of X called the **Free minimal spanning forest** (FMSF). See Alexander (1995) and BLPS (1999c) for more information on minimal spanning forests, and their connections to percolation.

The FMSF F_X on X may be constructed directly, as follows: Label the edges of X by i.i.d. random variables $\{U_e\}$, uniform in $[0, 1]$. Then remove any edge e that has the highest label in some finite cycle. In other words, e is retained in F_X iff there is no path between the endpoints of e consisting of edges with labels lower than U_e. It is clear that F_X has no cycles a.s. Moreover, all connected components of F_X are infinite a.s., since for any finite set of vertices K, the edge e that has the lowest label among the edges connecting K to its complement, must be in F_X.

Newman and Stein (1996) conjectured that the FMSF in $\mathbb{Z}^d$ is disconnected if d is large. This is still open, but Theorem 1.1 implies the following.

COROLLARY 2.1. *Let X be a nonamenable connected graph X that has a quasi-transitive unimodular automorphism group G, and suppose that the graph Y has an automorphism group H with an infinite orbit. Then the* FMSF *on $X \times Y$ has infinitely many components a.s.*

Proof. Denote by $N(F)$ the number of components of the FMSF F on $X \times Y$. Since $N(F)$ is invariant under the ergodic action of $G \times H$, it is a.s. constant. Theorem 1.1 ensures this constant is greater than 1. Moreover, if $N(F) < \infty$, then choosing uniformly one of

the $N(F)$ components of F would yield a $G \times H$-invariant measure on subtrees of $X \times Y$, contradicting Corollary 1.2. Therefore, $N(F) = \infty$ a.s. ∎

We note that the *uniform* spanning forest in a nonamenable product $X \times Y$ as in Theorem 1.1 is known to have infinitely many components, see BLPS (1999b).

The next corollary concerns minimal spanning trees in certain finite graphs. Nevertheless, we do not know any finitistic proof.

COROLLARY 2.2. *Let $T_d(n)$ be a finite tree with root ρ, where every vertex has degree $d \geq 2$, except the vertices at the maximal distance n from the root, that have degree 1. Let Γ_n be the minimal spanning tree in the product graph $T_d(n) \times T_b(n)$, determined by i.i.d. uniform labels on the edges. Denote by L_n the sum of the distances in Γ_n from (ρ, ρ) to its neighbors in $T_d(n) \times T_b(n)$. If $d \geq 3$, then the random variables L_n are not tight, i.e., $\sup_{M>1} \inf_n \mathbf{P}[L_n \leq M] < 1$.*

Proof. Denote the d-regular infinite tree by T_d, and consider the finite trees $T_d(n)$ as embedded in T_d, with the roots identified. Label the edges of $T_d \times T_b$ by uniform variables $\{U_e\}_{e \in E}$. The resulting minimal spanning trees on $T_d(n) \times T_b(n)$ converge almost surely to the FMSF F in $T_d \times T_b$.

If the variables L_n were tight, then passage to the limit would imply that the distance in F between any two vertices that are adjacent in $T_d \times T_b$ is finite a.s., so F is connected, contradicting Corollary 2.1. ∎

§3. Background on amenability.

Let G be a locally compact group, endowed with a left-invariant Haar measure. A linear functional M on $L^\infty(G)$ is called a **mean** if it maps the constant function $\mathbf{1}$ to 1 and $M(f) \ge 0$ for $f \ge 0$. If $f \in L^\infty(G)$ and $g \in G$, we write $L_g f(h) := f(gh)$. We call a mean M **invariant** if $M(L_g f) = M(f)$ for all $f \in L^\infty(G)$ and $g \in G$. Finally, we say that G is **amenable** if there is an invariant mean on $L^\infty(G)$. See Paterson (1988) for properties and characterizations of amenable groups, and the papers Soardi and Woess (1990), Salvatori (1992) and BLPS (1999a) for the relation between isoperimetric inequalities in a graph X and nonamenability of $\mathrm{Aut}(X)$.

We need the following variant of a method due to Adams and Lyons (1990).

LEMMA 3.1. *Let G be a locally compact group, that acts transitively on the countable set X, so that for every $x \in X$, the* **stabilizer** *$S_x := \{g \in G\,;\ g(x) = x\}$ is compact. We are given that G also acts by measure-preserving maps on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Suppose that for each $n \ge 1$ and $\omega \in \Omega$, an equivalence relation $R_n = R_n(\omega) \subset X \times X$ is given, so that for any $x_1, x_2 \in X$, the set $\{\omega\,;\ (x_1, x_2) \in R_n(\omega)\}$ is in $\mathcal{F}$. We assume that*

$$\forall g \in G, \qquad (x_1, x_2) \in R_n(\omega) \ \text{ iff } \ (gx_1, gx_2) \in R_n(g\omega)\,. \tag{3.1}$$

If for every $x \in X$ and $n \ge 1$, and for almost every $\omega \in \Omega$, the equivalence class $\mathcal{C}_n(x) := \{z \in X\,;\ (x, z) \in R_n\}$ is finite, and $\lim_n \mathbf{P}[(x, z) \in R_n] = 1$ for every $x, z \in X$, then G is amenable.

Proof. The hypothesis that G preserves $\mathbf{P}$, together with (3.1), gives

$$\mathbf{P}[(gx, gz) \in R_n] = \mathbf{P}[(x, z) \in R_n] \tag{3.2}$$

for all $x, z \in X$ and $g \in G$.

A linear functional M on $\ell^\infty(X)$ is called a *mean* if it maps the constant function $\mathbf{1}$ to 1 and $M(f) \ge 0$ for $f \ge 0$. For $f \in \ell^\infty(X)$ and $g \in G$, we write $L_g f(x) := f(gx)$. A mean M on X is **G-invariant** if $M(L_g f) = M(f)$ for all $f \in \ell^\infty(X)$.

Fix $o \in X$. For each n, define a mean M_n on X by $M_n(f) := \mathbf{E}[\sum_{x \in \mathcal{C}_n(o)} f(x)/|\mathcal{C}_n(o)|]$. By (3.2),

$$M_n(L_g f) = \mathbf{E}[\sum_{g^{-1}z \in \mathcal{C}_n(o)} f(z)/|\mathcal{C}_n(o)|] = \mathbf{E}[\sum_{z \in \mathcal{C}_n(go)} f(z)/|\mathcal{C}_n(go)|]\,.$$

Therefore

$$M_n(L_g f) - M_n(f) \le 2\|f\| \mathbf{P}[(o, go) \notin R_n] \to 0 \text{ as } n \to \infty\,.$$

Consequently, any weak* limit point M_* of $\{M_n\}$ is a G-invariant mean on X. Next, given $f \in L^\infty(G)$, define $\overline{f} \in \ell^\infty(X)$ by $\overline{f}(x) := \mu(S_o)^{-1} \int_{ho=x} f(h)\, d\mu(h)$, where μ is Haar measure. It is easy to check that $\overline{L_g f} = L_g \overline{f}$, so that an invariant mean M on G may be defined by $M(f) := M_*(\overline{f})$. ∎

§4. Spanning trees not confined to graphs.

The following theorem extends Theorem 1.1 and Theorem 1.3.

THEOREM 4.1. *Let G be a locally compact group, that acts on the countable set X, so that for each $x \in X$, the stabilizer $S_x \subset G$ is compact. Let H be a group that acts on the countable set Y so that*

$$\textit{for any finite subset } Y' \textit{ of } Y, \textit{ there exists } h \in H \textit{ satisfying } h(Y') \cap Y' = \emptyset. \tag{4.1}$$

If there exists a $G \times H$-invariant probability measure on the ensemble of spanning trees in $X \times Y$, then G is amenable.

Remark. Let H be a group of permutations of the countable set Y. As noted by B. Weiss and the referee, the hypothesis (4.1) on H holds if and only if all H-orbits in Y are infinite. See Newman (1976).

The next lemma is obvious if G is a group of graph automorphisms.

LEMMA 4.2. *Suppose that the locally compact group G acts on the countable set X with compact stabilizers. Then there exist sets $\{B(x,\ell)\,;\, x \in X, \ell \geq 1\}$ in X, such that for any $x \in X$ and $g_1 \in G$, we have*

$$g_1 B(x,\ell) = B(g_1 x, \ell) \ \text{ and } \ \cup_{\ell \geq 1} B(x,\ell) = X\,. \tag{4.2}$$

Moreover, for any $x_1 \in X$ and $\ell \geq 1$, the set $\{x_2 \in X\,;\, B(x_1,\ell) \cap B(x_2,\ell) \neq \emptyset\}$ is finite.

Proof. Fix $x_0 \in X$, and finite sets $\{X_\ell\}_{\ell \geq 1}$ such that $\cup_{\ell \geq 1} X_\ell = X$. Define

$$B(x,\ell) := \{gz\,;\, z \in X_\ell,\, gx_0 = x\}\,.$$

This is a finite set, since the sets $\{g\,;\, gx_0 = x, gz = w\}$ with $z \in X_\ell$ and $w \in B(x,\ell)$, form an open cover of the compact set $\{g\,;\, gx_0 = x\}$. The properties in (4.2) are immediate. Finally, for any $w \in X$,

$$\{x_2 \in X\,;\, w \in B(x_2,\ell)\} = \bigcup_{z \in X_\ell} \{gx_0\,;\, gz = w\}$$

is a finite set. Taking the union of these sets over $w \in B(x_1,\ell)$ completes the proof of the lemma. ∎

Proof of Theorem 4.1. Let $\Upsilon := X \times Y$ and denote by $\Omega \subset \{0,1\}^{\Upsilon\times\Upsilon}$ the set of indicator functions of spanning trees in Υ. In particular, $\omega(u,v) = \omega(v,u)$ for any $\omega \in \Omega$ and $u, v \in \Upsilon$. Our hypothesis is that that there is a $G \times H$-invariant probability measure $\mathbf{P}$ on Ω.

For $\omega \in \Omega$, denote by $\Gamma(\omega)$ the corresponding spanning tree, *i.e.*, the set of unordered pairs $\{u, v\} \subset \Upsilon$ such that $\omega(u,v) = 1$. Let $\{Y_n\}_{n\geq 0}$ be an increasing sequence of finite sets, such that $Y_0 = \{y_0\}$ and $\cup_n Y_n = Y$. For each n, pick $h_n \in H$ such that $h_n(Y_n) \cap Y_n = \emptyset$, and denote $y_n = h_n(y_0)$. Recall the sets $B(x,\ell)$ from Lemma 4.2; the parameter $\ell = \ell(n)$ will be specified below.

Fix an orbit X_o of G in X. For $\omega \in \Omega$ and $n \geq 1$, define the equivalence relation $R_n = R_n(\omega)$ on X_o, by letting $(x_1, x_2) \in R_n$ iff

(i) the path in $\Gamma(\omega)$ from (x_1, y_0) to (x_2, y_0) is contained in $X \times Y_n$, and

(ii) the path in $\Gamma(\omega)$ from (x_1, y_n) to (x_2, y_n) is contained in $X \times h_n(Y_n)$;

(iii) the path in $\Gamma(\omega)$ from (x_1, y_0) to (x_1, y_n) is contained in $B(x_1, \ell) \times Y$;

(iv) the path in $\Gamma(\omega)$ from (x_2, y_0) to (x_2, y_n) is contained in $B(x_2, \ell) \times Y$.

For fixed x_1, x_2 in X_o, the events in (i) and (ii) above have the same probability, which tends to 1 as $n \to \infty$. By choosing $\ell = \ell(n)$ large enough, we can ensure that the events in (iii) and (iv) have probability at least $1 - 1/n$. for x_1, x_2 in X_o. Therefore $\lim_n \mathbf{P}\Big((x_1, x_2) \in R_n\Big) = 1$ for any x_1, x_2 in X_o. The invariance relation (3.1) is easily checked, so in order to apply Lemma 3.1, we just need to verify that for every $x_1 \in X_o$, the equivalence class $\mathcal{C}_n(x_1)$ is a.s. finite.

Suppose that $(x_1, x_2) \in R_n$ and consider the "cycle" obtained by concatenating the following paths in $\Gamma(\omega)$:

$$(x_1, y_0) \to (x_1, y_n) \to (x_2, y_n) \to (x_2, y_0) \to (x_1, y_0)\,. \tag{4.3}$$

Since $\Gamma(\omega)$ is a tree, every edge that is traversed in (4.3), must be traversed an even number of times. Parts (i) and (ii) in the definition of R_n imply that the first edge in the $\Gamma(\omega)$-path $(x_1, y_0) \to (x_1, y_n)$ that exits $X \times Y_n$, must also occur in the $\Gamma(\omega)$-path $(x_2, y_n) \to (x_2, y_0)$. Therefore, by parts (iii) and (iv) of that definition, for $\ell = \ell(n)$ we have $B(x_1, \ell) \cap B(x_2, \ell) \neq \emptyset$. Thus Lemma 4.2 ensures that $\mathcal{C}_n(x_1)$ is a.s. finite. We have verified all the hypotheses of Lemma 3.1 (with X_o in place of X), so G is amenable.

∎

Acknowledgement. We are grateful to Itai Benjamini, Russ Lyons and Oded Schramm for numerous discussions that led to this work. Russ Lyons suggested that a unimodularity assumption in a previous version of this note could be dispensed with. We also thank the referee, Scot Adams and Benjy Weiss for useful remarks.

REFERENCES

ADAMS, S. (1988). Indecomposability of treed equivalence relations. *Israel J. Math.* **64**, 362–380.

ADAMS, S. (1994). Indecomposability of equivalence relations generated by word-hyperbolic groups. *Topology* **33**, 785–798.

ADAMS, S. and LYONS, R. (1991). Amenability, Kazhdan's property and percolation for trees, groups and equivalence relations, *Israel J. Math.* **75**, 341–370.

ADAMS, S. and SPATZIER, R. J. (1990). Kazhdan groups, cocycles and trees. *Amer. J. Math.* 112 (1990), 271–287.

ALEXANDER, K. S. (1995). Percolation and minimal spanning forests in infinite graphs, *Ann. Probab.* **23**, 87–104.

BENJAMINI, I., LYONS, R., PERES, Y., and SCHRAMM, O. (1999a). Group-invariant percolation on graphs, *Geom. Funct. Anal.* **9**, 29–66.

BENJAMINI, I., LYONS, R., PERES, Y. and SCHRAMM, O. (1999b). Uniform spanning forests, *Ann. Probab.*, to appear.

BENJAMINI, I., LYONS, R., PERES, Y. and SCHRAMM, O. (1999c). Minimal spanning forests, *in preparation.*

GABORIAU, D. (1998). Coût des relations d'équivalence et des groupes. *Preprint.*

HÄGGSTRÖM, O. (1997). Infinite clusters in dependent automorphism invariant percolation on trees, *Ann. Probab.* **25**, 1423–1436.

NEWMAN, C. M. and STEIN, D. L. (1996). Ground-state structure in a highly disordered spin-glass model, *J. Statist. Phys.* **82**, 1113–1132.

NEWMAN, P. M. (1976). The structure of finitary permutation groups, *Arch. Math.* (Basel) **27**, 3–17.

ORNSTEIN, D. S. and WEISS, B. (1987). Entropy and isomorphism theorems for actions of amenable groups, *J. d'Analyse* **48**, 1–141.

PATERSON, A. L. T. (1988). *Amenability.* American Mathematical Soc., Providence.

SALVATORI, M. (1992). On the norms of group-invariant transition operators on graphs, *J. Theor. Probab.* **5**, 563–576.

SOARDI, P. M. and WOESS, W. (1990). Amenability, unimodularity, and the spectral radius of random walks on infinite graphs, *Math. Z.* **205**, 471–486.

TROFIMOV, V. I. (1985). Groups of automorphisms of graphs as topological groups, *Math. Notes* **38**, 717–720.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WI 53706

`pemantle@math.wisc.edu`

INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY, GIVAT RAM, JERUSALEM 91904, ISRAEL AND DEPARTMENT OF STATISTICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA

`peres@math.huji.ac.il`

`http://www.ma.huji.ac.il/~peres/`